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# A Note on *a*-Weyl's Theorem

Young Min Han[1]

*Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Korea*



and

Slaviša V. Djordjević

*Department of Mathematics, Faculty of Philosophy, University of Niš, Ćirila and Metodija 2, 18000 Niš, Yugoslavia*
E-mail: slavdj@archimed.filfak.ni.ac.yu

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If $T$ or $T^*$ is log-hyponormal then for every $f \in H(\sigma(T))$, Weyl's theorem holds for $f(T)$, where $H(\sigma(T))$ denotes the set of all analytic functions on an open neighborhood of $\sigma(T)$. Moreover, if $T^*$ is $p$-hyponormal or log-hyponormal or $M$-hyponormal then for every $f \in H(\sigma(T))$, $a$-Weyl's theorem holds for $f(T)$. Also it is shown that if we let $A := \{T \in B(H) : T^* \text{ is log-hyponormal}\}$ then the approximate point spectrum, the essential approximate point spectrum, and the Browder essential approximate point spectrum are continuous in $A$. 

*Key Words:* log-hyponormal operators; $M$-hyponormal operators; Weyl spectrum; Weyl's theorem; $a$-Weyl's theorem; $a$-Browder's theorem; Browder's theorem.

## 1. INTRODUCTION

Throughout this note let $B(H)$ and $K(H)$ denote respectively the algebra of bounded linear operators and the ideal of compact operators acting on an infinite dimensional separable Hilbert space $H$. If $T \in B(H)$ write $N(T)$ and $R(T)$ for the null space and range of $T$; $\alpha(T) = \dim N(T)$; $\beta(T) = \dim(H/clR(T))$; $\sigma(T)$ for the spectrum of $T$; $\sigma_a(T)$ for the

[1] Current address: Department of Mathematics, University of Iowa, 14 MacLean Hall, Iowa City, IA 52242-1419. E-mail: yhan@math.uiowa.edu.



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approximate point spectrum of $T$; $\pi_0(T)$ for the set of eigenvalues of $T$; $\pi_{00}(T)$ for the isolated points of $\sigma(T)$ which are eigenvalues of finite multiplicity; and $\pi_{00}^a(T)$ for the isolated points of $\sigma_a(T)$ which are eigenvalues of finite multiplicity.

An operator $T \in B(H)$ is called *Fredholm* if it has closed range with finite dimensional null space and its range of finite co-dimension. The *index* of a Fredholm operator $T \in B(H)$ is given by

$$\operatorname{ind}(T) = \alpha(T) - \beta(T).$$

An operator $T \in B(H)$ is called *Weyl* if it is Fredholm of index zero. An operator $T \in B(H)$ is called *Browder* if it is Fredholm "of finite ascent and descent": equivalently [11, Theorem 7.9.3] if $T$ is Fredholm and $T - \lambda I$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\omega(T)$, and the Browder spectrum $\sigma_b(T)$ of $T \in B(H)$ are defined by [10, 11]

$$\begin{aligned}\sigma_e(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\};\\ \omega(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\};\\ \sigma_b(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\},\end{aligned}$$

evidently

$$\sigma_e(T) \subseteq \omega(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \operatorname{acc} \sigma(T),$$

where we write acc $K$ for the accumulation points of $K \subseteq \mathbb{C}$. We say that *Weyl's theorem holds for $T \in B(H)$ if there is equality*

$$\sigma(T) \setminus \omega(T) = \pi_{00}(T), \tag{1.1}$$

*and that Browder's theorem holds for $T \in B(H)$ if there is equality*

$$\omega(T) = \sigma_b(T). \tag{1.2}$$

We consider the sets

$$\begin{aligned}\Phi_+(H) &= \{T \in B(H) : R(T) \text{ is closed and } \alpha(T) < \infty\};\\ \Phi_-(H) &= \{T \in B(H) : R(T) \text{ is closed and } \alpha(T^*) < \infty\};\\ \Phi_+^-(H) &= \{T \in B(H) : T \in \Phi_+(H) \text{ and } \operatorname{ind}(T) \leq 0\}.\end{aligned}$$

By definition, $\sigma_{le}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_+(H)\}$ is the left essential spectrum, $\sigma_{re}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_-(H)\}$ is the right essential spectrum, $\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_+^-(H)\}$ is the essential approximate point spectrum, and $\sigma_{ab}(T) = \cap\{\sigma_a(T + K) : TK = KT \text{ and } K \in K(H)$ is the Browder essential approximate point spectrum.

We say that *a-Weyl's theorem holds for* $T \in B(H)$ *if there is equality*

$$\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}^a(T), \tag{1.3}$$

*and that a-Browder's theorem holds for* $T \in B(H)$ *if there is equality*

$$\sigma_{ea}(T) = \sigma_{ab}(T). \tag{1.4}$$

It is known [7, 9, 12] that if $T \in B(H)$ then we have

$a$-Weyl's theorem $\Rightarrow$ Weyl's theorem $\Rightarrow$ Browder's theorem;

$a$-Weyl's theorem $\Rightarrow$ $a$-Browder's theorem $\Rightarrow$ Browder's theorem.

Weyl [19] has shown that the equality (1.1) holds for hermitian operators. Weyl's theorem has been extended from hermitian operators to hyponormal operators and to Toeplitz operators [6], and to several classes of operators including seminormal operators [2, 3]. Rakočević [15] has shown that the equality (1.3) holds for cohyponormal operators. Recently S. V. Djordjević and D. S. Djordjević [7] showed that if $T^*$ is quasihyponormal then $T$ obeys $a$-Weyl's theorem.

An operator $T \in B(H)$ is said to be *p-hyponormal* if

$$(T^*T)^p \geq (TT^*)^p,$$

and an operator $T \in B(H)$ is said to be *log-hyponormal* if

$$T \text{ is invertible and } \log(T^*T) \geq \log(TT^*).$$

An operator $T \in B(H)$ is said to be *M-hyponormal* if there exists a positive real number $M$ such that

$$M\|(T-z)x\| \geq \|(T-z)^*x\| \qquad \text{for all } z \in \mathbb{C} \textit{ and } \text{for all } x \in H.$$

In this paper we show that if $T^*$ is $p$-hyponormal or log-hyponormal or $M$-hyponormal then $a$-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$, where $H(\sigma(T))$ denotes the set of all analytic functions on an open neighborhood of $\sigma(T)$.

## 2. PRELIMINARY RESULTS

Let $T \in B(H)$ and let $f \in H(\sigma(T))$, where $H(\sigma(T))$ denotes the set of all analytic functions on an open neighborhood of $\sigma(T)$. It is well known that the inclusion $\sigma_{ea}(f(T)) \subset f(\sigma_{ea}(T))$ holds [16] and it is known that for $\sigma_{ab}$ the spectral mapping theorem holds [16]. A sufficient condition for the spectral mapping theorem for $\sigma_{ea}$ can be given in terms of the set

$$A(H) = \{S \in B(H) : \operatorname{ind}(S - \lambda)\operatorname{ind}(S - \mu) \geq 0 \\ \text{for all } \lambda, \mu \in \mathbb{C} \setminus \sigma_{le}(S)\}.$$

THEOREM 2.1. *If $T \in B(H)$ obeys a-Browder's theorem, then the following statements are equivalent:*

(1) $T \in A(H)$.

(2) $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$ *for every* $f \in H(\sigma(T))$.

(3) *a-Browder's theorem holds for* $f(T)$ *for every* $f \in H(\sigma(T))$.

*Proof.* (1) $\Leftrightarrow$ (2). Suppose that $T \in A(H)$. Since $\sigma_{ea}(f(T)) \subset f(\sigma_{ea}(T))$, it is sufficient to show that $f(\sigma_{ea}(T)) \subset \sigma_{ea}(f(T))$. Let

$$f(T) - \lambda = c(T - \alpha_1)(T - \alpha_2) \cdots (T - \alpha_n)g(T), \tag{2.1.1}$$

where $c, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ and $g(T)$ is invertible. If $\lambda \notin \sigma_{ea}(f(T))$, then $f(T) - \lambda \in \Phi_+(H)$ and $\operatorname{ind}(f(T) - \lambda) \leq 0$. Since the operators on the right side of (2.1.1) commute, $T - \alpha_i \in \Phi_+(H)$ for all $i = 1, 2, \ldots, n$. If $\operatorname{ind}(T - \lambda) \leq 0$ for all $\lambda \in \Phi_+(T)$, then $\operatorname{ind}(T - \alpha_i) \leq 0$ for all $i = 1, 2, \ldots, n$, where $\Phi_+(T) = \{\lambda \in \mathbb{C} : T - \lambda \in \Phi_+(H)\}$. So $\lambda \notin f(\sigma_{ea}(T))$. If $\operatorname{ind}(T - \lambda) \geq 0$ for all $\lambda \in \Phi_+(T)$, then $\operatorname{ind}(T - \alpha_i) \geq 0$ for all $i = 1, 2, \ldots, n$. So $\operatorname{ind}(f(T) - \lambda) = \operatorname{ind}(T - \alpha_1)(T - \alpha_2) \cdots (T - \alpha_n) = \sum_{i=1}^{n} \operatorname{ind}(T - \lambda_i) \geq 0$. Therefore $\operatorname{ind}(f(T) - \lambda) = 0$. But $\operatorname{ind}(T - \alpha_i) \geq 0$ for all $i = 1, 2, \ldots, n$; hence $\operatorname{ind}(T - \alpha_i) = 0$ for all $i = 1, 2, \ldots, n$. Therefore $T - \alpha_i \in \Phi_+^-(H)$ for all $i = 1, 2, \ldots, n$; hence $\lambda \notin f(\sigma_{ea}(T))$. Conversely, suppose that $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$ for every $f \in H(\sigma(T))$. Assume to the contrary that $T \notin A(H)$. Then there exist $\alpha_1, \alpha_2 \in \Phi_+(T)$ such that $\operatorname{ind}(T - \alpha_1) < 0$ and $\operatorname{ind}(T - \alpha_2) > 0$. Let $m = -\operatorname{ind}(T - \alpha_1)$ and $n = \operatorname{ind}(T - \alpha_2)$. Put $f(z) = (z - \alpha_1)^{n+1}(z - \alpha_2)^m$. Then $f(T) = (T - \alpha_1)^{n+1}(T - \alpha_2)^m \in \Phi_+(H)$ and $\operatorname{ind}(f(T)) = (n + 1)\operatorname{ind}(T - \alpha_1) + m \cdot \operatorname{ind}(T - \alpha_2) = (n + 1)(-m) + mn = -m < 0$. So $0 \notin \sigma_{ea}(f(T))$. But $\alpha_2 \in \sigma_{ea}(T)$; hence $0 = f(\alpha_2) \in f(\sigma_{ea}(T)) = \sigma_{ea}(f(T))$. This is a contradiction. Hence $T \in A(H)$.

(2) $\Leftrightarrow$ (3). If $a$-Browder's theorem holds for $f(T)$, then $\sigma_{ea}(f(T)) = \sigma_{ab}(f(T))$. So

$$\sigma_{ea}(f(T)) = \sigma_{ab}(f(T)) = f(\sigma_{ab}(T)) = f(\sigma_{ea}(T)),$$

and hence $f(\sigma_{ea}(T)) = \sigma_{ea}(f(T))$. Conversely, if $f(\sigma_{ea}(T)) = \sigma_{ea}(f(T))$ then

$$\sigma_{ab}(f(T)) = f(\sigma_{ab}(T)) = f(\sigma_{ea}(T)) = \sigma_{ea}(f(T)).$$

Therefore $a$-Browder's theorem holds for $f(T)$. ∎

An operator $T \in B(H)$ is called *approximate-isoloid* if every isolated point of $\sigma_a(T)$ is an eigenvalue of $T$ and an operator $T \in B(H)$ is called *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of $T$. Clearly, if $T$ is approximate-isoloid then it is isoloid. However, the converse is not true. Consider the following example: let $T = T_1 \oplus T_2$, where $T_1$ is the unilateral shift on $l_2$ and $T_2$ is an injective quasinilpotent on $l_2$. Then $\sigma(T) = \{z \in \mathbb{C} : |z| \leq 1\}$ and $\sigma_a(T) = \{z \in \mathbb{C} : |z| = 1\} \cup \{0\}$. Therefore $T$ is isoloid but is not approximate-isoloid.

THEOREM 2.2. *If $T \in B(H)$ obeys a-Weyl's theorem and it is approximate-isoloid, then the following statements are equivalent*:

(1) $T \in A(H)$.

(2) $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$ *for every* $f \in H(\sigma(T))$.

(3) *a-Weyl's theorem holds for* $f(T)$ *for every* $f \in H(\sigma(T))$.

*Proof.* (1) $\Leftrightarrow$ (2). See the proof of Theorem 2.1.

(2) $\Leftrightarrow$ (3). Suppose that $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$ for every $f \in H(\sigma(T))$. Since $T$ obeys $a$-Weyl's theorem, $\sigma_{ea}(T) = \sigma_{ab}(T)$ [7, 9]. So $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T)) = f(\sigma_{ab}(T)) = \sigma_{ab}(f(T))$. Therefore $a$-Browder's theorem holds for $f(T)$. Now we claim that if $\lambda \in \pi_{00}^a(f(T))$, then $R(f(T) - \lambda)$ is closed. Let $\lambda \in \pi_{00}^a(f(T))$. Then $\lambda$ is an isolated point of $f(\sigma_a(T)$ and $0 < a(f(T) - \lambda) < \infty$. Since $\lambda$ is an isolated point of $f(\sigma_a(T))$, if $\alpha_i \in \sigma_a(T)$, then $\alpha_i$ is an isolated point of $\sigma_a(T)$. Therefore by (2.1.1), $0 < \alpha(T - \alpha_i) < \infty$ because $T$ is approximate-isoloid. Since $T$ obeys $a$-Weyl's theorem, $\alpha_i \notin \sigma_{ea}(T)$ for all $i = 1, 2, \ldots, n$. Therefore $f(T) - \lambda$ has closed range. By [9, Theorem 3.8], $f(T)$ obeys $a$-Weyl's theorem. Conversely, suppose $a$-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$. Then $\sigma_{ea}(f(T)) = \sigma_{ab}(f(T))$. But $a$-Weyl's theorem holds for $T$; hence $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$. ∎

## 3. MAIN RESULTS

For log-hyponormal operators, the Weyl spectrum obeys the spectral mapping theorem.

THEOREM 3.1. *If $T$ or $T^*$ is log-hyponormal then*

$$\omega(f(T)) = f(\omega(T)) \quad \text{for every } f \in H(\sigma(T)). \tag{3.1.1}$$

*Proof.* If $T$ or $T^*$ is log-hyponormal then $\operatorname{ind}(T - \lambda I) \leq 0$ or $\operatorname{ind}(T - \lambda(I) \geq 0$ for each $\lambda \in \mathbb{C} \setminus \sigma_e(T)$, respectively [4]. Thus we have

$$\operatorname{ind}(T - \lambda I) \cdot \operatorname{ind}(T - \mu I) \geq 0 \quad \text{for each } \lambda, \mu \in \mathbb{C} \setminus \sigma_e(T).$$

Therefore by an argument of Harte and Lee [12],

$$\omega(f(T)) = f(\omega(T)) \quad \text{when } f \text{ is a polynomial.}$$

Thus the equality (3.1.1) for $f \in H(\sigma(T))$ follows at once from an argument of Oberai [14, Theorem 2]. ∎

It was shown in [4, Theorem 7] that Weyl's theorem holds for log-hyponormal operators. We can prove more:

THEOREM 3.2. *If $T$ or $T^*$ is log-hyponormal then Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

*Proof.* Suppose $T$ is log-hyponormal. Then by [4, Theorem 7] Weyl's theorem holds for $T$. Remembering [13, Lemma] that if $T$ is isoloid then

$$f(\sigma(T) \setminus \pi_{00}(T)) = \sigma(f(T)) \setminus \pi_{00}(f(T)) \quad \text{for every } f \in H(\sigma(T)),$$

it follows from Theorem 3.1

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma(T) \setminus \pi_{00}(T)) = f(\omega(T)) = \omega(f(T)),$$

which implies that Weyl's theorem holds for $f(T)$. Now suppose $T^*$ is log-hyponormal. We first show that Weyl's theorem holds for $T$. Suppose $\lambda \in \pi_{00}(T)$. Then $\lambda$ is an isolated point of $\sigma(T)$ and $0 < \alpha(T-\lambda) < \infty$. Since $\sigma(T) = (\sigma(T^*))^-$, $\bar{\lambda}$ is an isolated point of $\sigma(T^*)$. But since $T^*$ is log-hyponormal, it follows that $\bar{\lambda} \in \pi_{00}(T^*)$. Since $T^*$ obeys Weyl's theorem, $\bar{\lambda} \in \sigma(T^*) \setminus \omega(T^*)$. Therefore $T - \lambda$ is Weyl, and so $\lambda \in \sigma(T) \setminus \omega(T)$. Conversely, suppose $\lambda \in \sigma(T) \setminus \omega(T)$. Observe $\sigma(T^*) = (\sigma(T))^-$ and $\omega(T^*) = (\omega(T))^-$. So $\bar{\lambda} \in \sigma(T^*) \setminus \omega(T^*)$, and hence $\bar{\lambda} \in \pi_{00}(T^*)$. Therefore $\lambda$ is an isolated point of $\sigma(T)$, and hence $\lambda \in \pi_{00}(T)$. Thus Weyl's theorem holds for $T$. Since if $T^*$ is log-hyponormal then $T$ is isoloid, it follows from Theorem 3.1 and the same argument as above that Weyl's theorem holds for $f(T)$. ∎

If $T^*$ is $p$-hyponormal or log-hyponormal, then we can prove more:

THEOREM 3.3. *If $T^*$ is $p$-hyponormal or log-hyponormal, then $f(T)$ obeys $a$-Weyl's theorem for every $f \in H(\sigma(T))$.*

*Proof.* Suppose $T^*$ is $p$-hyponormal. Then $T$ obeys $a$-Weyl's theorem [8]. Now we claim that $T$ is approximate-isoloid. Let $\lambda$ be an isolated point of $\sigma_a(T)$ and suppose $\lambda \notin \pi_0(T)$. Then $T - \lambda$ is one to one. Since $T^*$ is $p$-hyponormal, $\sigma(T) = \sigma_a(T)$. So $\lambda$ is an isolated point of $\sigma(T)$, and hence $T - \lambda$ is Weyl. Hence $T - \lambda$ is invertible. This is a contradiction. Thus $T$ is approximate-isoloid. Therefore by Theorem 2.2, $f(T)$ obeys $a$-Weyl's theorem. Now suppose $T^*$ is log-hyponormal. We first show that

$T$ obeys $a$-Weyl's theorem. Suppose $\lambda \in \pi_{00}^{a}(T)$. Then $\lambda$ is an isolated point of $\sigma_a(T)$ and $0 < \alpha(T-\lambda) < \infty$. So there exists a $\delta > 0$ such that if $0 < |\mu - \lambda| < \delta$, then $T - \mu$ is bounded below. But since $T^*$ is log-hyponormal it follows that $(T-\mu)^*$ is one to one. Thus $T - \mu$ is invertible, and so $\lambda$ is an isolated point of $\sigma(T)$. Therefore $\overline{\lambda}$ is an isolated point of $\sigma(T^*)$, and hence $\overline{\lambda} \in \pi_{00}(T^*)$. Since $T^*$ obeys Weyl's theorem, we have that $\overline{\lambda} \notin \omega(T^*)$. Therefore $T - \lambda$ is Weyl, and hence $\lambda \notin \sigma_{ea}(T)$. Conversely, suppose $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$. Then $T - \lambda \in \Phi_+(H)$ with $\mathrm{ind}(T - \lambda) \leq 0$ and $\alpha(T-\lambda) > 0$. Since $T^*$ is log-hyponormal, $\beta(T-\lambda) \leq \alpha(T - \lambda)$. Therefore $\mathrm{ind}(T-\lambda) = \alpha(T-\lambda) - \beta(T-\lambda) \geq 0$, and hence $T - \lambda$ is Weyl. But by Theorem 3.2, $T$ obeys Weyl's theorem and hence $\lambda$ is an isolated point of $\sigma(T)$. Therefore $\lambda$ is an isolated point of $\sigma_a(T)$, and hence $\lambda \in \pi_{00}^{a}(T)$. Thus $T$ obeys $a$-Weyl's theorem. Now we claim that $T$ is approximate-isoloid. Let $\lambda$ be an isolated point of $\sigma_a(T)$. Then there exists a $\delta > 0$ such that if $0 < |\mu - \lambda| < \delta$, then $T - \mu$ is bounded below. Since $T^*$ is log-hyponormal we have that $T - \mu$ is invertible. So $\overline{\lambda}$ is an isolated point of $\sigma(T^*)$, and hence $\lambda \in \pi_0(T)$. Therefore $T$ is approximate-isoloid. Let $f \in H(\sigma(T))$. We shall show that $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$. Since $\sigma_{ea}(f(T)) \subset f(\sigma_{ea}(T))$ for every $f \in H(\sigma(T))$ with no other restriction on $T$ [16], it suffices to show that $f(\sigma_{ea}(T)) \subset \sigma_{ea}(f(T))$. Suppose that $\lambda \notin \sigma_{ea}(f(T))$. Then $f(T) - \lambda \in \Phi_+(H)$ and

$$f(T) - \lambda = c(T-\alpha_1)(T-\alpha_2)\cdots(T-\alpha_n)g(T), \tag{3.3.1}$$

where $c, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ and $g(T)$ is invertible. Since the operators on the right side of (3.3.1) commute, $T - \alpha_i \in \Phi_+(H)$. Since $T^*$ is log-hyponormal, we have that $\mathrm{ind}(T-\alpha_i) \geq 0$. Therefore $\lambda \notin f(\sigma_{ea}(T))$, and hence $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$. It follows from Theorem 2.2 that $f(T)$ obeys $a$-Weyl's theorem. ∎

LEMMA 3.4. *Suppose that $T^* \in B(H)$ is $M$-hyponormal. Then $\alpha(T^* - \overline{\lambda}) \leq \alpha(T-\lambda)$ for all $\lambda \in \mathbb{C}$.*

*Proof.* Let $\lambda \in \mathbb{C}$ and suppose $x \in N(T^* - \overline{\lambda})$. Then $T^*x = \lambda^* x$. Since $T^*$ is $M$-hyponormal, there exists a positive real number $M$ such that $M\|(T^* - z)y\| \geq \|(T^*-z)^*y\|$ for all $z \in \mathbb{C}$ and for all $y \in H$. Therefore $Tx = \lambda x$, and so $x \in N(T-\lambda)$. Therefore $\alpha(T^* - \overline{\lambda}) \leq \alpha(T-\lambda)$. ∎

Our next theorem is an improvement of the result of Arora and Kumar [1] that Weyl's theorem holds for $M$-hyponormal operators:

THEOREM 3.5. *If $T$ or $T^*$ is $M$-hyponormal, then $f(T)$ obeys Weyl's theorem for every $f \in H(\sigma(T))$.*

*Proof.* If $T$ is $M$-hyponormal, then by [1, Theorem 4] $T$ obeys Weyl's theorem. But since $T$ is $M$-hyponormal, $\omega(f(T)) = f(\omega(T))$ for every

$f \in H(\sigma(T))$. It follows from [14, Theorem 1] that $f(T)$ obeys Weyl's theorem. Now suppose $T^*$ is $M$-hyponormal. Since $M$-hyponormality is translation-invariant, it suffices to show that $0 \in \pi_{00}(T) \Leftrightarrow 0 \in \sigma(T) \setminus \omega(T)$. Suppose $0 \in \pi_{00}(T)$. Then 0 is an isolated point of $\sigma(T)$ and $0 < \alpha(T) < \infty$. Since $\sigma(T) = ((\sigma(T^*))^-$, 0 is an isolated point of $\sigma(T^*)$. But since $T^*$ is $M$-hyponormal, it follows from Lemma 3.4 that $\alpha(T^*) < \infty$. Since $T^*$ is isoloid, $0 \in \pi_{00}(T^*)$. But since Weyl's theorem holds for $T^*$, hence $0 \in \sigma(T^*) \setminus \omega(T^*)$. Therefore $T$ is Weyl, and hence $0 \in \sigma(T) \setminus \omega(T)$. Conversely, suppose $0 \in \sigma(T) \setminus \omega(T)$. Observe that $\sigma(T^*) = (\sigma(T))^-$ and $\omega(T^*) = (\omega(T))^-$. So $0 \in \sigma(T^*) \setminus \omega(T^*)$, and hence $0 \in \pi_{00}(T^*)$. Therefore 0 is an isolated point of $\sigma(T)$, and hence $0 \in \pi_{00}(T)$. Thus Weyl's theorem holds for $T$. Since if $T^*$ is $M$-hyponormal then $T$ is isoloid, by the same argument as above, Weyl's theorem holds for $f(T)$. ∎

If $T^*$ is $M$-hyponormal, then we can prove more:

THEOREM 3.6. *If $T^*$ is $M$-hyponormal, then $f(T)$ obeys $a$-Weyl's theorem for every $f \in H(\sigma(T))$.*

*Proof.* We first show that $T$ obeys $a$-Weyl's theorem. Since $M$-hyponormality is translation-invariant, it suffices to show that $0 \in \pi_{00}^a(T) \Leftrightarrow 0 \in \sigma_a(T) \setminus \sigma_{ea}(T)$. Suppose $0 \in \pi_{00}^a(T)$. Then 0 is an isolated point of $\sigma_a(T)$ and $0 < \alpha(T) < \infty$. Since 0 is an isolated point of $\sigma_a(T)$, there exists a $\delta > 0$ such that if $0 < |\mu| < \delta$, then $T - \mu$ is bounded below. But $T^*$ is $M$-hyponormal; hence by Lemma 3.4, $(T - \mu)^*$ is one to one. So $T - \mu$ is invertible, and so 0 is an isolated point of $\sigma(T)$. Therefore 0 is an isolated point of $\sigma(T^*)$, and hence by Lemma 3.4, $0 \in \pi_{00}(T^*)$. Since $T^*$ obeys Weyl's theorem, $0 \notin \omega(T^*)$. Therefore $T$ is Weyl, and hence $0 \notin \sigma_{ea}(T)$. Conversely, suppose $0 \in \sigma_a(T) \setminus \sigma_{ea}(T)$. Then $T \in \Phi_+(H)$ with $\mathrm{ind}(T) \leq 0$ and $\alpha(T) > 0$. Since $T^*$ is $M$-hyponormal, by Lemma 3.4, $\alpha(T^*) \leq \alpha(T)$. Therefore $\mathrm{ind}(T) = \alpha(T) - \alpha(T^*) \geq 0$, and hence $T$ is Weyl. But by Theorem 3.5, $T$ obeys Weyl's theorem; hence 0 is an isolated point of $\sigma(T)$. Therefore 0 is an isolated point of $\sigma_a(T)$, and hence $0 \in \pi_{00}^a(T)$.

Now we claim that $T$ is approximate-isoloid. Let 0 be an isolated point of $\sigma_a(T)$. Then there exists a $\delta > 0$ such that if $0 < |\mu| < \delta$, then $T - \mu$ is bounded below. But $T^*$ is $M$-hyponormal; hence by Lemma 3.4, $T - \mu$ is invertible. So 0 is an isolated point of $\sigma(T^*)$. But since $T^*$ is $M$-hyponormal, $T^*$ is isoloid. So $\alpha(T^*) > 0$, and so by Lemma 3.4, $0 \in \pi_0(T)$. Therefore $T$ is approximate-isoloid. Let $f \in H(\sigma(T))$. We shall show that $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$. Since $\sigma_{ea}(f(T)) \subset f(\sigma_{ea}(T))$ for every $f \in H(\sigma(T))$ with no other restriction on $T$ [16], it suffices to show that $f(\sigma_{ea}(T)) \subset \sigma_{ea}(f(T))$. Suppose that $\lambda \notin \sigma_{ea}(f(T))$. Then $f(T) - \lambda \in \Phi_+(H)$ and

$$f(T) - \lambda = c(T - \alpha_1)(T - \alpha_2) \cdots (T - \alpha_n) g(T), \tag{3.6.1}$$

where $c, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ and $g(T)$ is invertible. Since the operators on the right side of (3.6.1) commute, $T - \alpha_i \in \Phi_+(H)$. Since $T^*$ is $M$-hyponormal, we have that by Lemma 3.4, $\operatorname{ind}(T - \alpha_i) \geq 0$. Therefore $\lambda \notin f(\sigma_{ea}(T))$, and hence $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$. Hence by Theorem 2.2, $f(T)$ obeys $a$-Weyl's theorem for every $f \in H(\sigma(T))$. ∎

## 4. BERBERIAN SPECTRA

Suppose that $T \in B(H)$ is reduced by each of its eigenspaces. If $\mathfrak{M}$ is the closed linear span of the eigenspaces $N(T - \lambda I)$ ($\lambda \in \pi_0(T)$), then $\mathfrak{M}$ reduces to $T$. Let $T_1 := T|\mathfrak{M}$ and $T_2 := T|\mathfrak{M}^\perp$. Then we have [3, Proposition 4.1]

(i) $T_1$ is a normal operator with pure point spectrum;

(ii) $\pi_0(T_1) = \pi_0(T)$;

(iii) $\sigma(T_1) = \operatorname{cl} \pi_0(T_1)$;

(iv) $\pi_0(T_2) = \varnothing$.

In this case, Berberian [3, Definition 5.2] defined

$$\tau(T) := \sigma(T_2) \cup \operatorname{acc} \pi_0(T) \cup \pi_{0i}(T).$$

We shall call $\tau(T)$ the *Berberian spectrum* of $T$. Berberian has shown that $\tau(T)$ is a nonempty compact subset of $\sigma(T)$. The following theorem shows the relation of essential approximate point spectra, Browder essential approximate point spectra, and Berberian spectra:

THEOREM 4.1. *If $T \in B(H)$ is reduced by each of its eigenspaces, then*

$$\sigma_{ab}(T) = \sigma_{ea}(T) \subset \tau(T). \tag{4.1.1}$$

*Proof.* Let $\mathfrak{M}$ be the closed linear span of the eigenspaces $N(T - \lambda I)$ ($\lambda \in \pi_0(T)$) and write

$$T_1 := T|\mathfrak{M} \qquad \text{and} \qquad T_2 := T|\mathfrak{M}^\perp.$$

From the preceding arguments it follows that $T_1$ is normal, $\pi_0(T_1) = \pi_0(T)$, and $\pi_{0f}(T_2) = \varnothing$. For (4.1.1) it will be shown that

$$\sigma_{ea}(T) \subset \tau(T); \tag{4.1.2}$$

and

$$\sigma_{ab}(T) \subset \sigma_{ea}(T). \tag{4.1.3}$$

For the first inclusion of (4.1.2) suppose $\lambda \in \sigma(T) \setminus \tau(T)$. Then $T_2 - \lambda$ is invertible and $\lambda$ is an isolated point of $\pi_0(T_1)$. Since also $\pi_{0i}(T_1) \subset \tau(T)$, we have $\lambda \in \pi_{00}(T_1)$. But $T_1$ is normal; hence $\pi_{00}(T_1) = \pi_{00}^a(T_1)$. So $\lambda \in \pi_{00}^a(T_1)$. Since $a$-Weyl's theorem holds for $T_1$, $\lambda \notin \sigma_{ea}(T_1) = \sigma_{ab}(T_1)$. Therefore $\lambda \notin \sigma_{ea}(T)$. This proves that (4.1.2). For the inclusion (4.1.3) suppose $\lambda \in \sigma(T) \setminus \sigma_{ea}(T)$. Observe that if $H_1$ is a Hilbert space and if an operator $R \in B(H_1)$ satisfies the equality $\sigma_{le}(R) = \omega(R)$, then

$$\sigma_{ea}(R \oplus S) = \sigma_{ea}(R) \cup \sigma_{ea}(S)$$

for each Hilbert space $H_2$ and $S \in B(H_2)$.

Indeed if $\lambda \notin \sigma_{ea}(R) \cup \sigma_{ea}(S)$, then $R - \lambda \in \Phi_+^-(H_1)$ and $S - \lambda \in \Phi_+^-(H_1)$ with $\operatorname{ind}(R - \lambda) \le 0$ and $\operatorname{ind}(S - \lambda) \le 0$. So $(R - \lambda) \oplus (S - \lambda) \in \Phi_+(H_1 \oplus H_2)$ and $\operatorname{ind}((R - \lambda) \oplus (S - \lambda)) \le 0$. Therefore $\lambda \notin \sigma_{ea}(R \oplus S)$. Conversely, suppose $\lambda \notin \sigma_{ea}(R \oplus S)$. Then $R - \lambda \in \Phi_+(H_1)$ and $S - \lambda \in \Phi_+(H_2)$. But $\sigma_{le}(R) = \omega(R)$; hence $\operatorname{ind}(R - \lambda) = 0$. Therefore $\operatorname{ind}(S - \lambda) \le 0$, and hence $\lambda \notin \sigma_{ea}(R) \cup \sigma_{ea}(S)$. Since $T_1$ is normal, applying the equality (4.1.4) to $T_1$ in place of $R$ gives that $\lambda \notin \sigma_{ea}(T_1) \cup \sigma_{ea}(T_2)$. But $T_1$ is normal; hence $\lambda \notin \sigma_{ab}(T_1)$. Since $T$ is reduced by each of its eigenspaces, by [16, Theorem 2.1], $\lambda \notin \sigma_{ab}(T_2)$. Therefore $\lambda \notin \sigma_{ab}(T)$. This proves (4.1.3) and completes the proof. ∎

As applications of Theorem 4.1 we will give several corollaries below.

COROLLARY 4.2. *If $T \in B(H)$ is reduced by each of its eigenspaces, then $a$-Browder's theorem holds for $T$.*

*Proof.* If $T$ is reduced by each of its eigenspaces, then by Theorem 4.1, $\sigma_{ea}(T) = \sigma_{ab}(T)$. Therefore $a$-Browder's theorem holds for $T$. ∎

COROLLARY 4.3. *If $T$ or $T^*$ is $p$-hyponormal, then $a$-Browder's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

*Proof.* Suppose $T$ is $p$-hyponormal. Then $T$ is reduced by each of eigenspaces. So by Corollary 4.2, $T$ obeys $a$-Browder's theorem. But $T$ is reduced by each of eigenspaces; hence $\operatorname{ind}(T - \lambda) \le 0$ for all $\lambda \in \Phi_+(T)$. Therefore $T \in A(H)$, and it follows from Theorem 2.1 that $a$-Browder's theorem holds for $f(T)$. If $T^*$ is $p$-hyponormal, then by Theorem 3.2, $a$-Weyl's theorem holds for $f(T)$. Therefore $a$-Browder's theorem holds for $f(T)$. ∎

COROLLARY 4.4. *If $T$ or $T^*$ is $M$-hyponormal, then $a$-Browder's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

*Proof.* Suppose $T$ is $M$-hyponormal. Then $T$ is reduced by each of eigenspaces. So by Corollary 4.2, $T$ obeys $a$-Browder's theorem. But since

$T$ is reduced by each of eigenspaces, $\operatorname{ind}(T-\lambda) \leq 0$ for all $\lambda \in \Phi_+(T)$. So $T \in A(H)$, and hence by Theorem 2.1, $a$-Browder's theorem holds for $f(T)$. If $T^*$ is $M$-hyponormal, then by Theorem 3.6, $a$-Weyl's theorem holds for $f(T)$. Therefore $a$-Browder's theorem holds for $f(T)$. ∎

COROLLARY 4.5. *If $T \in B(H)$ is reduced by each of its eigenspaces, then $\sigma_a(T) \setminus \sigma_{ea}(T) \subset \pi_{00}^a(T)$.*

*Proof.* This immediately follows from Theorem 4.1. ∎

COROLLARY 4.6. *If $T \in B(H)$ is reduced by each of its eigenspaces and is reduction approximate-isoloid, then $f(T)$ obeys a-Weyl's theorem for every $f \in H(\sigma(T))$.*

*Proof.* We first show that $a$-Weyl's theorem holds for $T$. By Corollary 4.5, it suffices to show that $\pi_{00}^a(T) \subset \sigma_a(T) \setminus \sigma_{ea}(T)$. Let $\lambda \in \pi_{00}^a(T)$. Since $T$ is reduced by its eigenspaces, $T$ can be represented as the following $2 \times 2$ operator matrix with respect to the decomposition $N(T-\lambda) \oplus N(T-\lambda)^{\perp}$:

$$T - \lambda = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}.$$

We shall show that $0 \notin \sigma_a(S)$. Assume to the contrary that $0 \in \sigma_a(S)$. Then 0 is an isolated point of $\sigma_a(S)$. But $T$ is reduction approximate-isoloid; hence $0 \in \pi_0(S)$. This is a contradiction. Therefore $0 \notin \sigma_a(S)$, and hence $T-\lambda$ has closed range. Hence $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$, and so $T$ obeys $a$-Weyl's theorem. Therefore by Theorem 2.2, $f(T)$ obeys $a$-Weyl's theorem. ∎

COROLLARY 4.7. *If $T \in B(H)$ is M-hyponormal such that $\sigma(T) = \sigma_a(T)$, then $T$ obeys a-Weyl's theorem.*

*Proof.* By Corollary 4.5, it suffices to show that $\pi_{00}^a(T) \subset \sigma_a(T) \setminus \sigma_{ea}(T)$. Since $M$-hyponormality is translation-invariant, it suffices to show that $0 \in \pi_{00}^a(T) \Rightarrow 0 \in \sigma_a(T) \setminus \sigma_{ea}(T)$. Suppose $0 \in \pi_{00}^a(T)$. Then $0 \in \pi_{00}(T)$ because $\sigma(T) = \sigma_a(T)$. But since $T$ obeys Weyl's theorem, hence $T$ is Weyl. So $0 \in \sigma_a(T) \setminus \sigma_{ea}(T)$. ∎

If $(G_n)$ is a sequence of compact subsets of $\mathbb{C}$, then by the definition, its limit inferior is $\liminf G_n = \{\lambda \in \mathbb{C}: \text{there are } \lambda_n \in G_n \text{ with } \lambda_n \to \lambda\}$ and its limit superior is $\limsup G_n = \{\lambda \in \mathbb{C}: \text{there are } \lambda_{n_k} \in G_{n_k} \text{ with } \lambda_{n_k} \to \lambda\}$. If $\liminf G_n = \limsup G_n$, then $\lim G_n$ is defined by this common limit. A mapping $\mathbf{F}$, defined on $B(H)$, whose values are compact subsets of $\mathbb{C}$, is said to be upper (lower) semi-continuous at $T$, provided that if $T_n \to T$ (in norm topology) then $\limsup \mathbf{F}(T_n) \subset \mathbf{F}(T)$ ($\mathbf{F}(T) \subset \liminf \mathbf{F}(T_n)$). If $\mathbf{F}$ is both upper and lower semi-continuous at $T$, then it is said to be continuous at $T$ and in this case $\lim \mathbf{F}(T_n) = \mathbf{F}(T)$.

Recently, Cho *et al*. showed that the spectrum is a continuous function in the set of all log-hyponormal operators [5]. From the fact, we can show that the essential approximate point spectrum, the Browder essential approximate point spectrum, and the approximate point spectrum are continuous in the set $A := \{T \in B(H) : T^* \text{ is log-hyponormal}\}$.

THEOREM 4.8. *If $T_n^*$, $T^*$ are log-hyponormal and $T_n \to T$, then* $\lim \sigma_{ea}(T_n) = \sigma_{ea}(T)$.

*Proof.* Since $\sigma_{ea}$ is upper semi-continuous, we have to show that $\sigma_{ea}(T) \subset \liminf \sigma_{ea}(T_n)$. Suppose $\lambda \notin \liminf \sigma_{ea}(T_n)$. Then there exists a subsequence $(\sigma_{ea}(T_{n_k}))$ of $(\sigma_{ea}(T_n))$ such that $d(\lambda, \sigma_{ea}(T_{n_k})) \geq \varepsilon$ for all $k$ and for some $\varepsilon > 0$. Without loss of generality we assume that $d(\lambda, \sigma_{ea}(T_n)) \geq \varepsilon$ for all $n$ and for some $\varepsilon > 0$. Then $T_n - \lambda \in \Phi_+^-(H)$ for all $n$, that is, $T_n - \lambda \in \Phi_+(H)$ and $\mathrm{ind}(T_n - \lambda) \leq 0$ for all $n$. If $x \in N(T_n^* - \overline{\lambda})$ then $T_n^*(x) = \overline{\lambda}x$. Since $T_n^*$ is log-hyponormal, $T_n x = \lambda x$. So $N(T_n^* - \overline{\lambda}) \subset N(T - \lambda)$, and hence $\mathrm{ind}(T_n - \lambda) \geq 0$. Therefore $T_n - \lambda$ is Weyl for all $n$. Since $T_n^*$, $T^*$ are log-hyponormal and $T_n^* \to T^*$, it follows from [5, Theorem 2.2] that $\sigma(T_n^*) \to \sigma(T^*)$. But since $T^*$ is log-hyponormal, hence Browder's theorem holds for $T^*$. So by [9, Theorem 2.2], $\omega(T_n^*) \to \omega(T^*)$. Hence $\omega(T) \subset \liminf \omega(T_n)$. But $T_n - \lambda$ is Weyl for all $n$; hence $\lambda \notin \liminf \omega(T_n)$. Therefore $\lambda \notin \sigma_{ea}(T)$. ∎

THEOREM 4.9. *If $T_n^*$, $T^*$ are log-hyponormal and $T_n \to T$, then* $\lim \sigma_{ab}(T_n) = \sigma_{ab}(T)$.

*Proof.* Suppose that $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$. Then $T - \lambda \in \Phi_+^-(H)$ and $0 < \alpha(T - \lambda) < \infty$. By the proof of Theorem 4.8, we get again $\alpha(T^* - \overline{\lambda}) \leq \alpha(T - \lambda) < \infty$. Since $T - \lambda \in \Phi_+^-(H)$, we have $T - \lambda$ is Weyl. So $\overline{\lambda} \notin \omega(T^*)$. But Weyl's theorem holds for log-hyponormal; hence $\overline{\lambda} \in \pi_{00}(T^*)$ and $\lambda$ is an isolated point of $\sigma(T)$. Now $\lambda$ is an isolated point of $\sigma_a(T)$. Therefore we get $\lambda \notin \sigma_{ab}(T)$. Hence $\sigma_{ea}(T) = \sigma_{ab}(T)$. Now we have that

$$\sigma_{ab}(T) = \sigma_{ea}(T) \subset \liminf \sigma_{ea}(T_n) \subset \liminf \sigma_{ab}(T_n).$$

Since $\sigma_{ab}$ is upper semi-continuous, $\lim \sigma_{ab}(T_n) = \sigma_{ab}(T)$. ∎

THEOREM 4.10. *If $T_n^*$, $T^*$ are log-hyponormal and $T_n \to T$, then* $\lim \sigma_a(T_n) = \sigma_a(T)$.

*Proof.* Since $\sigma_a$ is upper semi-continuous, it is sufficient to show that $\sigma_a(T) \subset \liminf \sigma_a(T_n)$. Let $\lambda \in \sigma_a(T)$. Now we consider two cases:

*Case I*. If $\lambda \in \sigma_{ea}(T)$, by Theorem 4.8 we have

$$\lambda \in \sigma_{ea}(T) \subset \liminf \sigma_{ea}(T_n) \subset \liminf \sigma_a(T_n).$$

*Case II.* If $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$ then $\lambda$ is an isolated point of $\sigma(T)$. So we have $\lambda \in \liminf \sigma(T_n)$. So there exists a sequence $(\lambda_n)$ in $\sigma(T_n)$ such that $\lambda_n \to \lambda$. If $\lambda_n \in \sigma(T_n) \setminus \sigma_a(T_n)$ then $\alpha(T_n - \lambda_n) = 0$. Since $T_n^*$ are log-hyponormal and $\beta(T_n - \lambda_n)^* = \alpha(T_n - \lambda_n) < \infty$, we have that $\beta(T_n - \lambda_n) = \alpha(T_n - \lambda_n)^* \leq \beta(T_n - \lambda_n)^* = \alpha(T_n - \lambda_n) = 0$. Therefore $\lambda_n \notin \sigma(T_n)$. This is contrary to the assumption that $\lambda_n \in \sigma(T_n) \setminus \sigma_a(T_n)$. Therefore we have $\lambda_n \in \sigma_a(T_n)$, and so $\lambda \in \liminf \sigma_a(T_n)$. ▮

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